1.1.2. Properties of Rings

1) **Uniqueness of Zero Elements:** Let z_1 and z_2 are two zero elements. (i.e., $a + z_1 = z_1 + a = a$, for all a, and also $a + z_2 = z_2 + a = a$ for all a.) Hence the zero element of a ring is unique.

   Then, $z_1 = z_1 + z_2 = z_2$

2) **Uniqueness of Additive Inverse:** Let b and c are both additive inverses of a. (i.e., $(a + b) = (b + a) = 0$ and $(a + c) = (c + a) = 0$, hence a unique zero element is here and this element is called as 0.) Then,
   $b = b + 0 = b + (a + c) = (b + a) + c = 0 + c = c$
   The additive inverse of an element a is unique.

3) **Cancellation Laws (Proposition):** According to right cancellation law, if $(a + x) = (b + x)$ is in a ring then $a = b$, also from left cancellation law if $(x + a) = (x + b)$, then $a = b$.

   Proof: Let us consider $(a + x) = (b + x)$ and $y = -x$. Then,
   $a = a + 0 = a + (x + y) = (a + x) + y = (b + x) + y = b + (x + y) = b + 0 = b$.

   Similarly other law is proved by using the commutativity of addition. These facts are under the cancellation laws.

Theorem 1: Property of Zero: In a ring (R, +, ·) $a \cdot 0 = 0 \cdot a = 0, a \in R$

Proof: According to left distributive law, we have

$a \cdot 0 + a \cdot 0 = a \cdot (0 + 0)$

$\because 0 + 0 = 0$ identity then $= a \cdot 0$

$\because 0$ is additive identity then $= a \cdot 0 + 0$

From Cancellation law we have $\Rightarrow a \cdot 0 = 0$

Similarly we can prove $0 \cdot a = 0$

Hence this is proved.

Theorem 2: Multiplication of Negative: In a ring (R, +, ·)

1) **$a(-b) = -(ab) = (-a) \cdot b$ where $a, b \in R$**
2) **$(-a)(-b) = a \cdot b$**

Proof:

1) The additive inverse of an element $b \in R$ is $-b \in R$ because (R, +) is an additive group.

   Now from distributive law, we have

   $a \cdot b + a(-b) = a(b + (-b))$ (Distributive law)

   From additive law, we have

   $= a \cdot 0$ (Additive law)

   $= 0$ (By proper of zero)

   $\Rightarrow$ Inverse of ab is $a(-b)$

   $\Rightarrow a(-b) = -(ab)$

   Thus we can show that, $(-a) \cdot b = -(ab)$

2) substituting $-b$ on the place of b on the above equation we get

   $(-a)(-b) = -[a(-b)]$

   Since $(-a)b = -ab$ then we have

   $= -[-(ab)]$ [Since $(-a)b = -ab$]

   $= ab$ [$\because (a^{-1})^{-1} = a$]

Theorem 3: Distributive Laws for Subtraction: In a ring (R, +, ·), $a, b, c \in R$

1) **$a \cdot (b - c) = a \cdot b - a \cdot c$ and**
2) **$(b - c) \cdot a = b \cdot a - c \cdot a$**

Proof:

1) We have $a \cdot (b - c) = a[b + (-c)]$
   From left distributive law, we have, $= a \cdot b + a(-c)$
   From **theorem 2**, we get, $= a \cdot b + [-(ac)] = a \cdot b - a \cdot c$
   In the same way on applying the right distribution law we can prove.

2) $(b - c)a = b \cdot a - c \cdot a$